Stabilization of linear higher derivative gravity with constraints

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Abstract

Linear instability in non-degenerate higher derivative theories, which is known as Ostrogradski's instability, can be removed by the addition of constraints. These constraints must reduce the dimension of original phase space. In other words, instabilities are removed only if constraints reduce the dynamical degrees of freedom(d.o.f) from original ones. Also, theories with curvature invariants such as $R_{\mu\nu}R^{\mu\nu}$, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, $C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}$, have the Ostrogradski's instability because they contain higher derivatives of the metric with respect to time.

I will start with a review about the stabilization of higher derivative gravity models. I consider the Lagrangian of the form $\mathcal{L} = \sqrt{-g}(R - 2\Lambda + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu})$. First, I give the second-order action for metric perturbations on a general background. Then, I focus on the Minkowski background, and demonstrate how the instabilities appear in each type of perturbations (i.e. scalar, vector, and tensor modes.) by constructing the Hamiltonian. I show that those instabilities can be removed by imposing constraints on the theory. Finally, I will give some comments on cosmological implications of the constrained theory.

1 Introduction

Non-degenerate higher derivative theories suffer from the Ostrogradski's instability. The statement of Ostrogradski's instability is that we cannot avoid appearing the ghost in non-degenerate higher derivative theories. However, several theories such as $f(R)$ gravity evade this instability while they contain higher derivatives of the metric with respect to time. This is because they are degenerate, which means that they are constrained. Unstable degrees of freedom of $f(R)$ and GR is removed by gauge constraint. In general, non-degenerate higher derivative theories are inevitably unstable.

We consider the following action

$$\begin{aligned} S &= \frac{M_{\rm Pl}^2}{2}\int d^4x\sqrt{-g} \\ &\quad \times \left[R - 2\Lambda + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\right] \ . \quad (1) \end{aligned}$$

First, we show that this theory is unstable, i.e. , the ghost will appear in this theory. Then, we stabilize the theory by adding suitable constraints which reduce the dimension of the phase space.

2 Higher derivative gravity: quadratic action

We consider the Lagrangian which contains the quadratic curvature invariant $R_{\mu\nu}R^{\mu\nu}$.

$$\frac{\mathcal{L}}{\sqrt{-g}} = \frac{M_{\rm Pl}^2}{2}\left[R - 2\Lambda + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}\right] \ . \quad (2)$$

Our first purpose is to calculate the action up to the quadratic order in the metric perturbation $h_{\mu\nu}$, which is given by

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \ . \quad (3)$$

We obtain Ricci tensor, Ricci scalar,and Einstein tensor in linear order as follows.

$$\begin{aligned} R^L_{\mu\nu} &= \frac{1}{2}\left(\bar{\nabla}_\rho\bar{\nabla}_\mu h^\rho{}_\nu + \bar{\nabla}_\rho\bar{\nabla}_\nu h^\rho{}_\mu \right. \\ &\qquad \left. - \Box h_{\mu\nu} + \bar{\nabla}_\mu\bar{\nabla}_\nu h\right) \ , \\ R^L &= \bar{g}_{\mu\nu}\delta R^L_{\mu\nu} - \bar{R}^{\mu\nu}h_{\mu\nu} \ , \\ G^L_{\mu\nu} &= \delta R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\delta R^L - \Lambda h_{\mu\nu} \ . \end{aligned}$$

Note that, the indices are raised and lowered by back ground metric $\bar{g}_{\mu\nu}$. Assuming that constant curvature background of either Minkowski, de Sitter, Anti-de Sitter, the Weyl tensor vanishes, we should thus compute the linear order term of weyl tensor to calculate the action up to quadratic order. Then, we obtain the Second order action,

$$\begin{aligned} S &= -\frac{M_{\rm Pl}^2}{4}\int d^4x\sqrt{-\bar{g}}h^{\mu\nu} \\ &\quad \Big[(1+8\alpha\Lambda+\frac{4}{3}\beta\Lambda)G^L_{\mu\nu} \\ &\quad +(\beta+2\alpha)(\bar{g}_{\mu\nu}\Box-\bar{\nabla}_\mu\bar{\nabla}_\nu+\Lambda\bar{g}_{\mu\nu})R^L \\ &\quad +\beta(\Box G^L_{\mu\nu}-\frac{2\Lambda}{3}\bar{g}_{\mu\nu}R^L)\Big] \ . \end{aligned} \tag{4}$$

In this equation, we omit δ so δR is denoted as R^L.

3 Metric perturbations around Minkowski background and instabilities

Metric perturbations are given by

$$\begin{aligned} ds^2 &= -(1+2A)dt^2+2(B_{,i}-S_i)dx^idt \\ &+ \left[(1-2\psi)\delta_{ij}+2E_{,ij}+2F_{(i,j)}+h_{ij}\right]dx^idx^j \ . \end{aligned}$$

Here, A, B, E, and ψ are scalar perturbations. S_i and F_i are vector perturbations and are transverse. h_{ij} is tensor perturbation and is transverse and traceless. We will consider each types of perturbations in the following sections.

3.1 Tensor perturbation : Helicity-2 sector

In this section, we consider the tensor perturbation, which has helicity-2. Using Eq. (4), (5), we obtain

$$\begin{aligned} S = \frac{M_{\rm Pl}^2}{2}\int d^4x\Big[\beta\left\{(\ddot{h}_{ij})^2+2\dot{h}^{ij}\nabla^2\dot{h}_{ij}+(\nabla^2h_{ij})^2\right\} \\ +(\dot{h}_{ij})^2+h^{ij}\nabla^2h_{ij}\Big] \ . \end{aligned} \tag{5}$$

Notice that only the term which is proportional to β survives. This action contains higher order derivative with respect to t. Let us discuss the instability of this system by constructing the Hamiltonian. The choice of canonical variables are

$$\begin{aligned} h^{(1)}_{ij} \equiv h_{ij} \quad &\leftrightarrow \quad \pi^{ij} = 2\dot{h}^{ij}+\beta(-2\dddot{h}^{ij}+4\nabla^2\dot{h}^{ij}) \ , \\ h^{(2)}_{ij} \equiv \dot{h}_{ij} \quad &\leftrightarrow \quad p^{ij} = 2\beta\ddot{h}^{ij} \ . \end{aligned}$$

Using these definitions, we can construct the Hamiltonian by Legendre transform:

$$\begin{aligned} H = \frac{M_{\rm Pl}^2}{2}\int d^3x\Big[\frac{1}{4\beta}p^{ij}p_{ij}+\pi^{ij}h^{(2)}_{ij}-2\beta h^{(2)}_{ij}\nabla^2h^{(2)}_{ij} \\ -h^{(2)}_{ij}h^{(2)ij}-\beta\nabla^2h^{(1)}_{ij}\nabla^2h^{(1)ij}-h^{(1)}_{ij}\nabla^2h^{(1)}_{ij}\Big] \ . \end{aligned} \tag{6}$$

This Hamiltonian is linearly dependent on π^{ij}, and therefore the Hamiltonian is not bounded from below, which means that the tensor perturbation of the theory is unstable.

3.2 Scalar perturbation : Helicity-0 sector

We can obtain the quadratic action of scalar perturbations as follows:

$$\begin{aligned} S &= \frac{M_{\rm Pl}^2}{2}\int d^4x\Big[(-6\dot{\Psi}^2-2\Psi\nabla^2\Psi+4\Psi\nabla^2\Phi) \\ &\quad +4(\beta+3\alpha)(3\ddot{\Psi}^2+4\dot{\Psi}\nabla^2\dot{\Psi}+2\ddot{\Psi}\nabla^2\Phi) \\ &\quad +2(3\beta+8\alpha)\left(\nabla^2\Psi\right)^2+2(\beta+2\alpha)\left(\nabla^2\Phi\right)^2 \\ &\quad -4(\beta+4\alpha)\nabla^2\Psi\nabla^2\Phi\Big] \ . \end{aligned} \tag{7}$$

Here, Ψ and Φ are the gauge invariant perturbations defined by

$$\Phi \equiv A+\dot{B}-\ddot{E} \ , \tag{8}$$

$$\Psi \equiv \psi \ . \tag{9}$$

Φ is non-dynamical since $\dot{\Phi}$ does not appear in the action. This means that variation with respect to Φ gives a constraint. The choice of canonical variables are

$$\Phi \equiv \Phi \quad \leftrightarrow \quad p_\Phi = 0 \ , \tag{10}$$

$$\Psi \equiv \Psi \quad \leftrightarrow \quad p_\Psi = \frac{\delta S}{\delta\dot{\Psi}} \ , \tag{11}$$

$$\chi \equiv \dot{\Psi} \quad \leftrightarrow \quad p_\chi = 8(\beta+3\alpha)(3\ddot{\Psi}+\nabla^2\Phi) \ . \tag{12}$$

Then, we have the Hamiltonian of the form

$$
\begin{aligned}
H &= \frac{M_{\rm Pl}^2}{2}\int d^3x \Bigg[p_\Psi \chi + \frac{p_\chi^2}{48(\beta+3\alpha)} - \frac{p_\chi \nabla^2\Phi}{3} \\
&\quad +(6\chi^2 + 2\Psi\nabla^2\Psi - 4\Psi\nabla^2\Phi) \\
&\quad -16(\beta+3\alpha)\chi\nabla^2\chi - 2(3\beta+8\alpha)(\nabla^2\Psi)^2 \\
&\quad 4(\beta+4\alpha)\nabla^2\Psi\nabla^2\Phi \frac{2\beta}{3}(\nabla^2\Phi)^2\Bigg] \ . \qquad (13)
\end{aligned}
$$

There is one constraint derived from variation with respect to Φ for this Hamiltonian. Substituting this constraint to the Hamiltonian, we obtain the reduced Hamiltonian of the form

$$
\begin{aligned}
H_R &= \frac{M_{\rm Pl}^2}{2}\int d^3x \\
&\quad \times \Bigg[p_\Psi \chi + \frac{1}{\beta}p_\chi \left\{1-(\beta+4\alpha)\nabla^2\right\}\Psi \\
&\quad + \frac{\beta+2\alpha}{16\beta(\beta+3\alpha)}p_\chi^2 + 6\chi^2 - 16(\beta+3\alpha)\chi\nabla^2\chi \\
&\quad + \frac{6}{\beta}\Psi^2 - (10 + \frac{48\alpha}{\beta})\Psi\nabla^2\Psi \\
&\quad + \frac{32\alpha(\beta+3\alpha)}{\beta}(\nabla^2\Psi)^2\Bigg] \ . \qquad (14)
\end{aligned}
$$

One can see that the instability appears also in the case of scalar perturbations.

4 Stabilization of the theory by adding constraints

In this section, we explain how the theory is stabilized. First, we discuss the tensor perturbation. Adding the constraint to Eq. (5) by using the Lagrange multiplier λ_{ij}, we obtain the quadratic action of the form

$$
\begin{aligned}
S &= \frac{M_{\rm Pl}^2}{2}\int d^4x \Big[\beta\Big\{(\ddot{h}_{ij} - \lambda_{ij})^2 + 2\dot{h}^{ij}\nabla^2\dot{h}_{ij} \\
&\quad +(\nabla^2 h_{ij})^2\Big\} + (\dot{h}_{ij})^2 + h^{ij}\nabla^2 h_{ij} \\
&\quad + 4\beta\lambda^{ij}\nabla^2 h_{ij}\Big] \ . \qquad (15)
\end{aligned}
$$

Then, we construct the Hamiltonian. The choice of canonical variables are

$$
h_{ij}^{(1)} \equiv h_{ij} \quad \leftrightarrow \quad \pi^{ij} = 2\dot{h}^{ij} + \beta(-2\dddot{h}^{ij} + 4\nabla^2\dot{h}^{ij} + 2\dot{\lambda}_{ij}) \ , \qquad (16)
$$

$$
h_{ij}^{(2)} \equiv \dot{h}_{ij} \quad \leftrightarrow \quad p^{ij} = 2\beta(\ddot{h}^{ij} - \lambda_{ij}) \ , \qquad (17)
$$

$$
\lambda_{ij} \equiv \lambda_{ij} \quad \leftrightarrow \quad p_\lambda^{ij} = 0 \ . \qquad (18)
$$

Using these definitions, the Hamiltonian is given of the form

$$
\begin{aligned}
H &= \frac{M_{\rm Pl}^2}{2}\int d^3x \Big[\pi^{ij}h_{ij}^{(2)} + \frac{1}{4\beta}p^{ij}p_{ij} \\
&\quad - h^{(1)ij}(\beta\nabla^2\nabla^2 + \nabla^2)h_{ij}^{(1)} - q^{ij}(1+2\beta\nabla^2)q_{ij} \\
&\quad + \lambda^{ij}(p_{ij} - 4\beta\nabla^2 h_{ij}^{(1)})\Big] \ . \qquad (19)
\end{aligned}
$$

Eq. (18) is the primary constraint. This constraint is satisfied only on the hypersurface in the phase space. We obtain the secondary constraints by implying that primary constraint holds for arbitrary time, i.e., Poisson bracket between primary constraint and the Hamiltonian should be 0. Then, we obtain a secondary constraint. Since this secondary constraint holds also for arbitrary time, we obtain the another secondary constraint, and so on. Finally, we obtain the secondary constraints as follows.

$$
\begin{aligned}
\phi_1 &: \quad p_{\lambda_{ij}} \approx 0 \ , & (20) \\
\phi_2 &: \quad p_{ij} - 4\beta\nabla^2 h_{ij} \approx 0 \ , & (21) \\
\phi_3 &: \quad \pi_{ij} - 2h_{ij}^{(2)} \approx 0 \ , & (22) \\
\phi_4 &: \quad 2(\beta\nabla^2\nabla^2 + \nabla^2)h_{ij}^{(1)} - \frac{1}{\beta}p_{ij} + 2(-1+2\beta\nabla^2)\lambda_{ij} \approx 0 \ . & (23)
\end{aligned}
$$

Here, $\approx$ is 'weak equality', which means that this equation is satisfied only on the hypersurface. Substituting these constraints to Eq. (15), we obtain the reduced Hamiltonian

$$
\begin{aligned}
H_R &= \frac{M_{\rm Pl}^2}{2}\int d^3x \Big[\frac{1}{4}\pi^{ij}(1-2\beta\nabla^2)\pi_{ij} \\
&\quad + h^{(1)ij}(-\nabla^2 + 3\beta\nabla^2\nabla^2)h_{ij}^{(1)}\Big] \ . \qquad (24)
\end{aligned}
$$

The reduced Hamiltonian is bounded from below, which means that the theory does not contain the ghost. We succeeded the stabilization of the theory.

Taking the same approach as tensor perturbation, we can discuss the scalar perturbations. We start with the action which contains the Lagrange multiplier λ. the action is given of the form

$$\begin{aligned} S &= \frac{M_{\rm Pl}^2}{2}\int d^4x \Big[(-6\dot{\Psi}^2 - 2\Psi\nabla^2\Psi + 4\Psi\nabla^2\Phi) \\ &+4(\beta+3\alpha)(3\ddot{\Psi}^2 + 4\dot{\Psi}\nabla^2\dot{\Psi} + 2\ddot{\Psi}\nabla^2\Phi) \\ &+2(3\beta+8\alpha)\left(\nabla^2\Psi\right)^2 + 2(\beta+2\alpha)\left(\nabla^2\Phi\right)^2 \\ &-4(\beta+4\alpha)\nabla^2\Psi\nabla^2\Phi + 32(\beta+3\alpha)\lambda\nabla^2\Psi \\ &+12(\beta+3\alpha)(\lambda^2 - 2\ddot{\Psi}\lambda - \frac{2}{3}\lambda\nabla^2\Phi) + C\lambda\Psi)\Big] . \end{aligned} \quad (25)$$

The choice of canonical variables are

$$\Phi \equiv \Phi \quad \leftrightarrow \quad p_\Phi = 0 \,, \quad (26)$$

$$\Psi \equiv \Psi \quad \leftrightarrow \quad p_\Psi = \frac{\delta S}{\delta\dot{\Psi}} \,, \quad (27)$$

$$\chi \equiv \dot{\Psi} \quad \leftrightarrow \quad p_\chi = 8(\beta+3\alpha)\left[3(\ddot{\Psi}-\lambda)+\nabla^2\Phi\right] \quad (28)$$

$$\lambda \equiv \lambda \quad \leftrightarrow \quad p_\lambda = 0 \,. \quad (29)$$

Eq. (26), (29) are primary constraints. We obtain the secondary constraints as follows.

$$\phi_1 \;:\; p_\Phi \approx 0 \,, \quad (30)$$

$$\phi_2 \;:\; p_\lambda \approx 0 \,, \quad (31)$$

$$\phi_3 \;:\; p_\chi - C\Psi - 32(\beta+3\alpha)\nabla^2\Psi \approx 0 \,, \quad (32)$$

$$\phi_4 \;:\; \nabla^2\left[\frac{p_\chi}{3} + 4\Psi - 4(\beta+4\alpha)\nabla^2\Psi + \frac{4\beta}{3}\nabla^2\Phi\right] \approx 0 \,, \quad (33)$$

$$\phi_5 \;:\; p_\Psi + (12+C)\chi \approx 0 \,, \quad (34)$$

$$\begin{aligned} \phi_6 \;:\; & \frac{(12+C)p_\chi}{24(\beta+3\alpha)} + 2(6+\alpha)\lambda + 32(\beta+3\alpha)\nabla^2\lambda \\ & +4(3\beta+8\alpha)\nabla^2\nabla^2\Psi - 4(\beta+4\alpha)\nabla^2\nabla^2\Phi \\ & -4\nabla^2\Psi - \frac{C}{3}\nabla^2\Phi \approx 0 \,. \end{aligned} \quad (35)$$

We can construct the Hamiltonian, and substitute these constraints. Then we obtain the reduced Hamiltonian of the form

$$\begin{aligned} H_R &= \frac{M_{\rm Pl}^2}{2}\int d^3x \\ & \Big[\frac{-p_\Psi}{(12+C)^2}\left\{(6+C) + 16(\beta+3\alpha)\nabla^2\right\} p_\Psi \\ & +\frac{1}{\beta}\left\{(6+C) + \frac{C^2(\beta+2\alpha)}{16(\beta+3\alpha)}\right\}\Psi^2 \\ & \left\{(22+\frac{48\alpha}{\beta}) + \frac{C}{\beta}(3\beta+4\alpha)\right\}\Psi\nabla^2\Psi \\ & +\frac{32(\beta+3\alpha)(\beta+\alpha)}{\beta}(\nabla^2\Psi)^2\Big] \,. \end{aligned} \quad (36)$$

This Hamiltonian is also bounded from below.

5 Conclusion and future outlook

First, we perturb the metric around the Minkowski space, and show that the theory is unstable by calculating the quadratic action to the metric fluctuations. Then, we removed those instability by adding suitable constraints.

The future task is improving the application to inflationary fluctuations in higher derivative gravity.

Reference

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